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Orbital Divergence and Relaxation in the Gravitational *N*-Body Problem

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One of the fundamental aspects of statistical behaviour in many-body systems is exponential divergence of neighbouring orbits, which is often discussed in terms of Liapounov exponents. Here we study this topic for the classical gravitational N-body problem. The application we have in mind is to old stellar systems such as globular star clusters, where $N \sim 10^6$, and so we concentrate on spherical, centrally concentrated systems with total energy $E < 0$. Hitherto no connection has been made between the time scale for divergence (denoted here by t_e) and the time scale on which the energies of the particles evolve because of two-body encounters (i.e., the two-body relaxation time scale, t_r), even though both may be calculated by similar considerations. In this paper we give a simplified model showing that divergence in phase space is initially roughly exponential, on a timescale proportional to the crossing time (defined as a mean time for a star to cross from one side of the system to another). In this phase $t_e \ll t_r$, if N is not too small (i.e., $N \gg 30$). After several e-folding times, the model shows that the divergence slows down. Thereafter the divergence (measured by the energies of the bodies) varies with time as $t^{1/2}$, on a timescale nearly proportional to the familiar two-body relaxation timescale, i.e., $t_e \sim t_r$ in this phase. These conclusions are illustrated by numerical results.

KEY WORDS: Gravity; few-body systems; relaxation processes; particle orbits.

1. INTRODUCTION

The classical gravitational N-body problem is defined by the equations

$$\ddot{\mathbf{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \tag{1}$$

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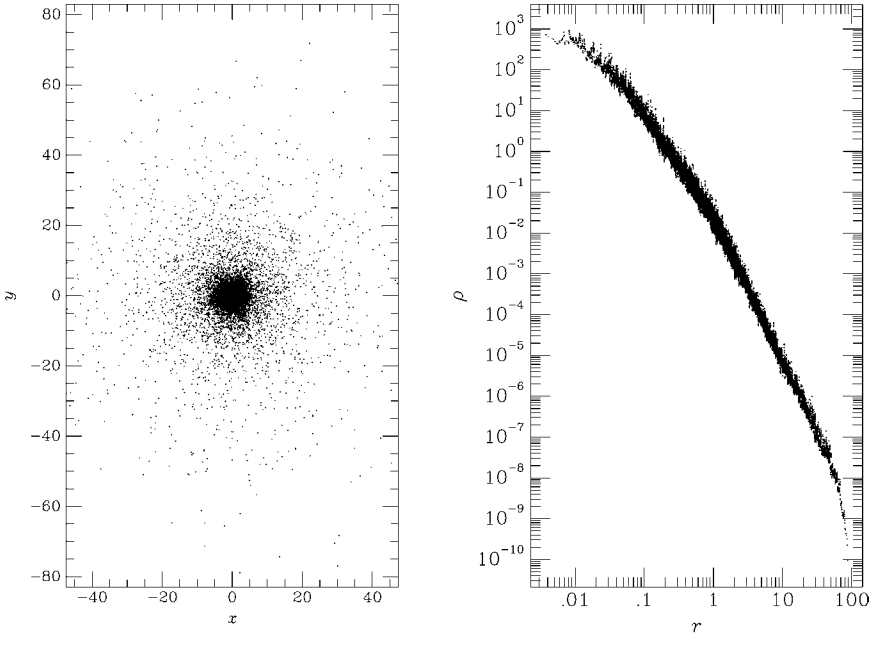

Fig. 1. Spatial distribution of bodies in a typical simulation. On left is a snapshot, and on the right is the numerically generated space density as a function of radius.

where $\mathbf{r}_i$ is the three-dimensional position vector of the ith star, m_i is its mass, and G is the universal constant of gravitation. We consider applications in which the total energy, E, in the barycentric frame is negative and the total angular momentum is negligible. Starting from a rather broad set of initial conditions, such solutions settle down into a roughly spherical distribution of bodies in approximate "dynamic equilibrium" (Fig. 1), i.e., the spatial distribution is nearly time-independent on the time scale of the orbital motions of the particles.

Early numerical integrations[1] with $N \leqslant 32$ showed that a small change in initial conditions led to a roughly exponential divergence of solutions (measured in $6N$-dimensional phase space), even though the spatial distribution of the bodies in the two solutions might be indistinguishable within statistical fluctuations. The timescale of divergence, t_e, was of order the crossing time, t_{cr}, defined in a certain conventional way as the time for a body with a typical speed to move a distance comparable to the size, R, of the spatial distribution of the particles.[2] Thus

$$t_{cr} \sim \frac{R}{V}, \tag{2}$$

where V is the root mean square speed of the particles. Later work[3–5] extended numerical results to larger N, and Goodman *et al.*[3] devised theoretical models confirming that t_e/t_{cr} is virtually independent of N.

One particular statistical specification of the initial conditions which has been studied is the Plummer model, which is often used in stellar dynamics for the study of relaxation and related processes. It is the stellar dynamical analogue of the $n = 5$ polytrope. For this model it has been found[4] that

$$t_e \simeq \frac{0.116 t_{cr}}{\ln(0.73 \ln N)}.$$

The functional form is suggested by a theoretical model,[3] and the coefficients are not thought to depend sensitively on the initial conditions. Therefore for large star clusters generally we have

$$t_e \sim 0.05 t_{cr}. \tag{3}$$

The theoretical models of Goodman *et al.*[3] dealt with the linear divergence of neighbouring solutions, when the separation in position of the ith body satisfies the variational equation

$$\delta\ddot{\mathbf{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{N} m_j \left(\frac{\delta\mathbf{r}_i - \delta\mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - \frac{(\delta\mathbf{r}_i - \delta\mathbf{r}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^5} (\mathbf{r}_i - \mathbf{r}_j) \right). \tag{4}$$

For practical purposes, however (e.g., for understanding the growth of errors in a numerical integration) the resulting roughly exponential growth quickly leads to separations so large that the linear approximation fails. In this contribution we develop the simplest model of divergence to account for the later, nonlinear growth of the separation between neighbouring solutions. We shall see that the time dependence changes from roughly exponential to roughly power-law, and that the timescale changes from roughly the crossing time to nearly the two-body relaxation timescale, t_r. This is the timescale on which the energies of the individual bodies vary significantly. Standard theory[2, 6] shows that

$$t_r \sim \frac{N}{\ln N} t_{cr} \tag{5}$$

for systems of the general kind considered here.

2. A MODEL OF DIVERGENCE

2.1. Linear Growth of Errors

In this section we introduce a toy model for the divergence of neighbouring orbits. Though it gives much insight into the physics of the problem, many details are omitted. In the first instance we apply it to the linear regime in which the approximate Eq. (4) are valid. In this regime more elaborate models have been constructed by Goodman *et al.*[3]

We make the following assumptions. As in the theory of two-body relaxation[2, 6] we assume that the trajectory of a particle is nearly rectilinear, except for occasional two-body encounters (Fig. 2). We suppose that the important encounters are in the small-angle scattering regime, such that $p \gg Gm/v^2$ where p is the impact parameter and v is the relative velocity of the two particles. In computing the effect of one encounter, we suppose we can treat the scatterer as fixed. We also suppose that successive encounters can be treated as if motion takes place on one plane, and that the difference between two orbits is measured by the difference in the impact parameter, δp. We assume that all particles have the same mass m. Finally, we suppose that the system is in virial equilibrium (see Binney and Tremaine[2]), which implies that

$$V^2 \sim \frac{GmN}{R}. \tag{6}$$

Here the symbol $\sim$ means "is of order," i.e., that the relation is approximate, and any numerical coefficient is ignored. Thus $v \sim V$, for example.

In the small-angle scattering regime the maximum acceleration of the moving particle is of order $\frac{Gm}{p^2}$ and the duration of the encounter is of order $\frac{p}{V}$. Thus the change in velocity is of order $\frac{Gm}{pV}$, and so the angular deflection is of order $\frac{Gm}{pV^2}$ (Fig. 2). After the scattered body has travelled

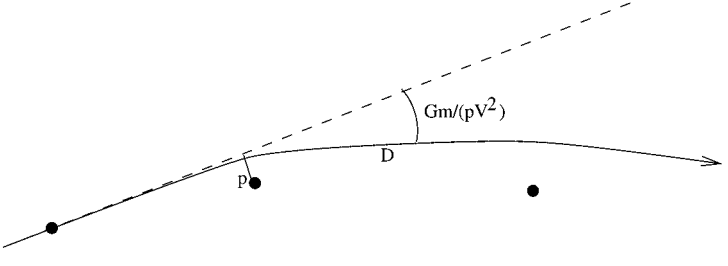

Fig. 2. Two successive encounters.

a further distance D to its next encounter, its spatial deflection is of order $\frac{GmD}{pV^2}$.

Now suppose the body had approached the first encounter on a parallel path at a slightly different impact parameter $p+\delta p$. Then, at the time of the second encounter, its position would have been displaced by a distance of order $\delta p+\frac{GmD}{p^2V^2}\delta p$. The first term is the displacement that would have occurred even in the absence of the first encounter. The second occurs because, if $\delta p>0$, the body has been deflected less by the first encounter. (The differential approximation used for this term is not valid unless $|\delta p| \ll p$; this is the approximation which restricts the present theory to the linear regime in which Eq. (4) is valid.) The total displacement measures the change in impact parameter at the second encounter. Hence the variation in p is multiplied by a factor of order $(1+\frac{GmD}{p^2V^2})$ per encounter.[3]

Now we consider the cumulative effect of several encounters within a restricted range of impact parameters around the value p, e.g., from $p/2$ to $2p$, but ignoring other encounters. We start at some time t and consider the effect of encounters in a subsequent interval Δt, chosen sufficiently large that several encounters occur within this interval. The actual number of such encounters is of order $\frac{\Delta tV}{D}$, and so the variation in the orbit is given[4] by

$$\delta r(t+\Delta t) \sim \delta r(t)\left(1+\frac{GmD}{p^2V^2}\right)^{\frac{\Delta tV}{D}}.$$

Also it is clear that $p^2Dn \sim 1$, where n is the number of particles per unit volume, and so

$$\delta r(t+\Delta t) \sim \delta r(t)\left(1+\frac{Gm}{p^4nV^2}\right)^{\Delta tVnp^2}. \tag{7}$$

It follows from the relation $n \sim \frac{N}{R^3}$ and Eq. (6) that

$$\delta r(t+\Delta t) \sim \delta r(t)\left(1+\frac{R^4}{p^4N^2}\right)^{\frac{\Delta t}{t_{cr}}\frac{Np^2}{R^2}}, \tag{8}$$

where we have used Eq. (2).

[3] In a fully three-dimensional treatment this becomes a matrix equation.

[4] We ignore two complications which tend to counteract each other: (i) the persistence of effects of early encounters, and (ii) partial cancellation of successive encounters by their vectorial character.

Encounters take place at a wide range of impact parameters p. Writing Eq. (8) as

$$\ln \delta r(t+\Delta t) - \ln \delta r(t) \sim \frac{\Delta t}{t_{cr}} N \frac{p^2}{R^2} \ln\left(1+\frac{R^4}{p^4 N^2}\right) \tag{9}$$

we see that those with $p \ll RN^{-1/2}$ are individually very effective but too rare to dominate, whereas those with $p \gg RN^{-1/2}$ lose out by being individually ineffective, despite being very numerous. Encounters at impact parameter $p \sim RN^{-1/2}$ are most effective cumulatively, and lead to exponential growth of the deviation δr, on a timescale of order t_{cr}.

Another way of seeing this is to sum the right hand side of Eq. (9) over all impact parameters $p \lesssim R$. Since this term represents the effect of encounters with impact parameters in a range near some value p, the summation can be accomplished by multiplying by dp/p and integrating. The result is that

$$\ln \delta r(t+\Delta t) - \ln \delta r(t) \sim \frac{\Delta t}{t_{cr}}.$$

This is equivalent to the result obtained by ignoring all encounters except those near $p \sim RN^{-1/2}$.

Many factors have been omitted from this simple model, including the distribution of velocities and density, and the curved orbits of bodies between encounters. Nevertheless, the results of more detailed models and numerical simulations, already quoted, confirm our basic result, except for a very weak N-dependence.

2.2. Nonlinear Growth of Separation

The above theory is valid as long as $\delta r \ll p$, and here we may take for p the impact parameter for the most effective encounters, i.e., $p \sim RN^{-1/2}$. Suppose we are interested in growth of errors in an N-body integration of Eq. (1), for a system which has been scaled so that $R \sim 1$. Then we may have $\delta r(0) \sim 10^{-16}$ for a double precision calculation, and so the linear approximation breaks down after about $30t_e$, i.e., between one and two t_{cr}, by Eq. (3).

Thereafter we suppose that encounters with impact parameters $p \ll \delta r$ are ineffective. Then we may estimate the growth of the separation of neighbouring orbits by substituting $p \sim \delta r(t)$ in Eq. (9), which gives

$$\ln \delta r(t+\Delta t) - \ln \delta r(t) \sim \frac{\Delta t}{t_{cr}} \frac{N \delta r(t)^2}{R^2} \ln\left(1+\frac{R^4}{N^2 \delta r(t)^4}\right).$$

We are in a regime where $\delta r(t) \gtrsim RN^{-1/2}$, and so we can approximate

$$\ln \delta r(t+\Delta t) - \ln \delta r(t) \sim \frac{\Delta t}{t_{cr}} \frac{R^2}{N\delta r(t)^2}. \tag{10}$$

Since the term on the right depends on t, we can no longer conclude that $\ln \delta r(t)$ increases linearly with t. To determine its time dependence we rewrite Eq. (10) as a differential equation, i.e.,

$$\frac{d}{dt} \ln \delta r(t) \sim \frac{1}{t_{cr}} \frac{R^2}{N\delta r(t)^2}.$$

Ignoring for the moment the distinction between "$\sim$" and "$=$," we obtain the solution

$$\delta r(t) = \left(\delta r(t_0)^2 + 2 \frac{t-t_0}{t_{cr}} \frac{R^2}{N} \right)^{1/2},$$

where t_0 is a constant, which may be interpreted as the time at which the growth of errors enters the nonlinear regime.

Well into the nonlinear regime we now see that $\delta r(t) \sim R(\frac{t}{Nt_{cr}})^{1/2}$. In order to interpret this result we shall estimate the difference in binding energy, ε, of the body between the two neighbouring solutions. Now $\varepsilon \sim \frac{GNm}{R}$, and we can estimate $\delta\varepsilon \sim \frac{GNm\delta r}{R^2}$. (We could obtain a similar estimate from consideration of the difference in velocity.) Hence $\frac{\delta\varepsilon}{\varepsilon} \sim (\frac{t}{Nt_{cr}})^{1/2}$. Now the two-body relaxation time, t_r, may be estimated by Eq. (5), and so $\frac{\delta\varepsilon}{\varepsilon} \sim (\frac{t}{t_r})^{1/2}$ if we ignore a logarithmic dependence on N.

3. DISCUSSION

Recall that we are considering two solutions of Eq. (1) starting with slightly different initial conditions. Suppose that we measure the separation of the two solutions by the separation in energy, $\delta\varepsilon$, of a typical body. What we have concluded is that, for at most a few crossing times, $\delta\varepsilon(t)$ grows exponentially, with an e-folding time comparable with t_{cr} itself. Thereafter $\delta\varepsilon(t)$ approaches a power law dependence, varying as $t^{1/2}$, on a timescale of the relaxation time.

The standard theory of relaxation tells us how ε (the energy of a given star) evolves on a *single* solution of the N-body equation. If we ignore variations of ε inside an encounter, ε performs a random walk on the timescale t_r, and the change in ε varies as $t^{1/2}$. (We here ignore the role of

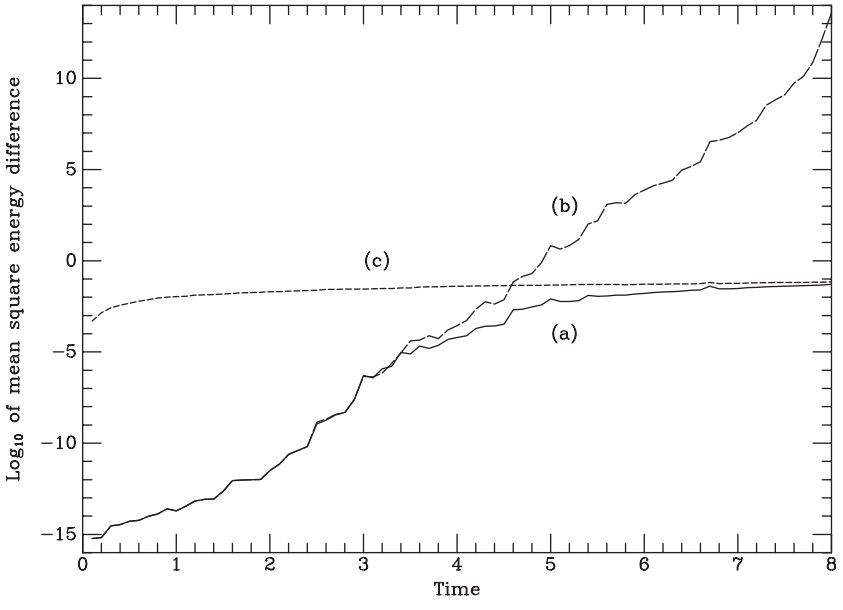

Fig. 3. Mean square energy difference in numerical integrations with $N = 256$, as a function of time. The meaning of the different curves is stated in the text. The results plotted are the mean of four independent runs. In the adopted units the crossing time is $2\sqrt{2}$.

"dynamical friction", which corresponds to the drift term in a Fokker–Planck description of the relaxation.[2, 6])

Figure 3 illustrates these points using data from numerical N-body integrations with $N = 256$. Two systems were integrated simultaneously using identical initial conditions except for a small difference in one coordinate of one particle. The solid curve (a) shows the mean square difference in the energies of the N particles.[5] The corresponding initial conditions were also used for simultaneous integration of the variational equations, and the long-dashed curve (b) shows the corresponding mean square variation of energy. This grows nearly exponentially, but is followed by (a) for only a limited time of order a crossing time. The short-dashed curve (c) shows the mean square difference between the initial energy and the energy at time t, again averaged over the N particles. This is caused by two-body relaxation. Evidently curve (a) departs from curve (b) around the point where the latter crosses curve (c), and then nearly follows (c). In this way

[5] Similar results have been presented by Merritt[7] for motion in the gravitational field of N *fixed* bodies.

we see that the growth of errors, which is exponential only in the linear regime, is consistent with the theory of two-body relaxation.

The exponential divergence slows down to a power-law growth because close encounters become increasingly ineffective. There is a geometric way of looking at this. Krylov[8] showed that the divergence could be understood as the behaviour of neighbouring geodesics on a certain manifold. As two neighbouring geodesics deviate further, their deviation is influenced less and less by the fine geometrical structure of the manifold across which they are proceeding.

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REFERENCES

1. R. H. Miller, Irreversibility in small stellar dynamical systems, *ApJ* **140**:250–256 (1964).
2. J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 1987).
3. J. Goodman, D. C. Heggie, and P. Hut, On the exponential instability of N-body systems, *ApJ* **415**:715–733 (1993).
4. M. Hemsendorf and D. Merritt, Instability of the gravitational N-body problem in the large-N limit (2002), astro-ph/0205538.
5. H. E. Kandrup and H. Smith, On the sensitivity of the N-body problem to small changes in initial conditions, *ApJ* **374**:255–65 (1991).
6. S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, 1942); also (Dover Publications, New York, 1960).
7. D. Merritt, Non-integrable galactic dynamics, in *The Restless Universe*, B. A. Steves and A. J. Maciejewski, eds. (SUSSP Publications, Edinburgh; and Institute of Physics Publishing, London, 2001).
8. N. S. Krylov, *Works on the Foundations of Statistical Physics* (Princeton University Press, Princeton, 1979).